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DETERMINATION OF MODE SHAPES FROM THE OPERATIONAL DEFLECTION SHAPE

Bor-Tsuen Wang
Department of Mechanical Engineering
National Pingtung University of Science and Technology
Pingtung 91207, Taiwan
e-mail: wangbt@mail.npust.edu.tw

Abstract

This paper presents the predictive algorithm to obtain the mode shapes of multiple degree-of-freedom (MDOF) system from the measured operational deflection shape (ODS). Assuming that the MDOF system is subjected to a known harmonic force acting at the known location. The steady state response of the system at each DOF is also harmonic and can be measured. The operational deflection shape defined as the amplitude of harmonic response at each DOF can then be obtained. The developed algorithm can be applied to determine the mode shape vectors for all modes, while only one set of ODS is available. This work first derives the harmonic response of the MDOF system due to a harmonic force. The optimization problem to determine the mode shape is then formulated. The objective function is defined as the sum of square value of error between the predicted and experimentally measured ODS at each DOF. The design variables can be identified as the mode shape vectors for all modes. The orthogonality of mode shape vectors with respect to mass and stiffness matrices are considered as constraint conditions. After the resolution of the optimization problem, the mode shape vectors can be determined. Several numerical examples are presented to demonstrate the feasibility of the predictive algorithm and shown successful in determining all of the mode shape vectors. The predictive model can also be extended to continuous structures and useful for structural fault diagnosis.

INTRODUCTION

Mode shape information is quite interested for structural fault diagnosis [1-5]. Traditionally, experimental modal analysis [6] by selecting proper actuator and sensors can be performed to obtain system frequency response functions. At least a column or a row of FRFs matrix must be measured and can then be processed to extract modal parameters, including natural frequencies, damping ratios and mode shapes. It is difficult for conventional modal testing, if the test structure is under operation. Many works have been dedicated to structural testing under operating conditions [7,8] or subject to natural excitation such as wind, traffic flow, etc.

As known, when a structure is harmonically excited in particular the excitation frequency equal or close to the structural natural frequency, the structural deflection shape reveals to be the mode shape of the corresponding natural mode [9]. The observation or measurement of the operational deflection shape (ODS) is another way to have the knowledge of the structural mode shape. The machinery in operating conditions is usually subjected harmonic excitation. The operational deflection shape is useful in predicting the structural fault [10-12]. Besides, the knowledge of structural mode shapes is also preferred. Although one can perform a complete modal testing to obtain the structural modal parameters, for some circumstance to conduct such a test may not be possible. However, the structural operational deflection shape due to harmonic input can be measured. The operational deflection shape is also strongly correlated to the structural mode shapes. The question is how one can get the mode shapes from the operational deflection shapes. This work, therefore, attempts to extract the mode shape information from single measurement of operational deflection shape.

Generally, system natural frequencies can be easily obtained from single mobility measurement. On the other hand, to determine the structural mode shapes requires conducting a complete modal testing and so forth to get a set of frequency response functions. Through the modal parameter extraction process, the mode shapes can then be obtained. The proposed algorithm in this work is based on the assumption of prior knowledge of system natural frequencies, the known force location and its amplitude as well as its excitation frequency. Performing single measurement of structural operational deflection shape makes the ODS available. The optimization problem to determine the mode shape vectors for all modes is formulated. The objective function is defined as the sum of square value of error between the predicted and experimentally measured ODS at each DOF. The design variables can be identified as the mode shape vectors for all modes. The orthonormality of mode shape vectors with respect to mass and stiffness matrices are considered as constraint conditions. A three-DOF and a four-DOF system are considered to demonstrate the feasibility of the proposed algorithm, respectively. Assuming that the system natural frequencies are known in prior, and the harmonic force is also known including its excitation frequency, location and amplitude. All of the mode shape vectors can then be successfully determined and converged to a satisfactory solution. The proposed algorithm can be extended to continuous structure system.

THEORETICAL ANALYSIS OF ODS FOR MDOF SYSTEM

Consider a proportionally, viscously damped multiple degree-of-freedom (MDOF) system that consists of n DOFs as follows:

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{f\} \tag{1}$$

where

$$[C]=\alpha[M]+\beta[K] \tag{2}$$

α and β are some constants.

A. Modal analysis:

By neglecting damping matrix and assuming zero force, one can formulate the eigenvalue problem for the undamped MDOF system as follows:

$$[K]\{X\}=\lambda[M]\{X\} \tag{3}$$

With the resolution of the above equation, n pairs of eigenvalues and their corresponding eigenvectors can be determined as follows:

$$\begin{cases}\lambda_r\,, r=1,2,\ldots\ldots,n \\ \{\psi\}_r\end{cases} \tag{4}$$

where

$$\lambda_r=\omega_r^2 \tag{5}$$

ω_r and $\{\psi\}_r$ are called the undamped natural frequency and mode shape vector for the system, respectively. They represent the natural modes of vibration for the system. The mass matrix normalized modal matrix can then be obtained and defined as follows:

$$[\Phi]=[\{\phi\}_1,\{\phi\}_2,\{\phi\}_3\ldots,\{\phi\}_n] \tag{6}$$

The relation of the orthonormality of the modal matrix can be shown in matrix form

$$[\Phi]^T[M][\Phi]=\mathrm{diag}[I] \tag{7}$$

$$[\Phi]^T[C][\Phi]=\mathrm{diag}[2\xi_r\omega_r] \tag{8}$$

$$[\Phi]^T[K][\Phi]=\mathrm{diag}[\omega_r^2] \tag{9}$$

where

$$\xi_r=\frac{\alpha}{2\omega_r}+\frac{\beta\omega_r}{2} \tag{10}$$

B. Harmonic response analysis:

For forced vibration analysis, the response can be assumed as follows:

$$\{x(t)\}=[\Phi]\{q(t)\}=\sum_{r=1}^{n}\{\phi\}_r q_r(t) \text{ or } x_j(t)=\sum_{r=1}^{n}\phi_{r,j}q_r(t) \tag{11}$$

By substituting Equation (11) into Equation (1), pre-multiplying $[\Phi]^T$ and adopting the orthonormality relationship of modal matrix, one can get the decoupled system equations as follows:

$$\mathrm{diag}[I]\{\ddot{q}\}+\mathrm{diag}[2\xi_r\omega_r]\{\dot{q}\}+\mathrm{diag}[\omega_r^2]\{q\}=\{N(t)\}=[\Phi]^T\{f\} \tag{12}$$

or

$$\ddot{q}_r+2\xi_r\omega_r\dot{q}_r+\omega_r^2q_r=N_r(t)=\sum_{k=1}^{n}\phi_{r,k}f_k \tag{13}$$

For the system subjected to harmonic excitation, the system response can also be harmonic. By assuming a harmonic force acting on the k-th location with amplitude F_k and excitation frequency ω_s, the force vector can be expressed as follows:

$$\{f\}=\begin{Bmatrix}0\\0\\\vdots\\F_k\\\vdots\\0\end{Bmatrix}e^{i\omega_s t} \tag{14}$$

Let $q_r = Q_k e^{i\omega_s t}$, $f_k = F_k e^{i\omega_s t}$ be substituted into Equation (13), one can derive:

$$Q_k = \frac{F_k \phi_{r,k}}{(\omega_r^2 - \omega_s^2) + i(2\xi_r \omega_r \omega_s)} \tag{15}$$

then from Equation (11),

$$x_j(t) = \sum_{r=1}^{n} \frac{F_k \phi_{r,k} \phi_{r,j}}{(\omega_r^2 - \omega_s^2) + i(2\xi_r \omega_r \omega_s)} e^{i\omega_s t} = X_j(\omega_s) e^{i\omega_s t} \tag{16}$$

The displacement vector can be obtained:

$$\{x\} = \{X(\omega_s)\} e^{i\omega_s t} = \begin{Bmatrix} X_1(\omega_s) \\ X_2(\omega_s) \\ \vdots \\ X_n(\omega_s) \end{Bmatrix} e^{i\omega_s t} \tag{17}$$

where $X_j(\omega_s)$ is the amplitude of the *j*-th location can be derived as follows

$$X_j(\omega_s) = \sum_{r=1}^{n} \frac{F_k \phi_{r,j} \phi_{r,k}}{(\omega_r^2 - \omega_s^2) + i(2\xi_r \omega_r \omega_s)} \tag{18}$$

$\{X(\omega_s)\}$ can be defined as the operational deflection shape (ODS) for the system subject to single harmonic excitaiton. It is noted that in conventional modal testing frequency response function is measured and processed to determine the modal parameters. In this work, ODS in Equation (18) is adopted to predict the mode shape vector

DETERMINATION OF MODE SHAPES FROM ODS

The main objective of this work is to determine the structural mode shapes from the measured ODS due to single harmonic excitation. The conceptual diagram to determine the mode shape is shown in Figure 1. The assumptions are summarized as follows:

1. Single ODS measurement data ($\{\hat{X}(\omega_s)\}$) is available. The structural natural frequencies (ω_r) and damping ratios (ξ_r) can easily be identified from the measured mobility data.
2. The excitation frequency (ω_s) and amplitude (F_k) of the harmonic force are also known.

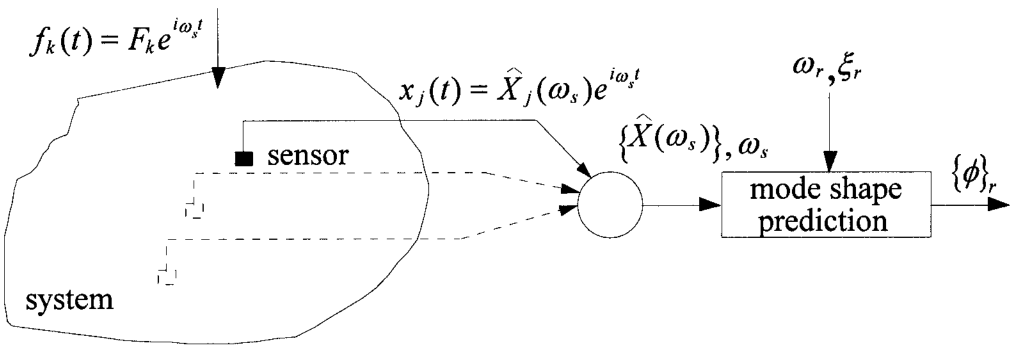

Figure 1. Conceptual diagram for mode shape prediction

3. The proportional, viscous damping is assumed.

The optimization problem can be formulated based upon the above assumptions. The verbal statement of the optimization problem can be described as follows. The objective is to determine the mode shape vector components that will minimize the errors between predicted and measured OSDs and satisfies the orthonormality conditions of modal matrix. The objective function is defined as the sum of square errors between predicted and measured ODS components. Design variables can then identified as the mode shape components. The orthonormality of modal matrix with respect to mass matrix and stiffness matrix are employed as the constraints. The mathematical model of the optimization problem can be expressed as follows:

Objective function:

$$\Phi = \sum_{j=1}^{n}\left[X_j(\omega_s) - \hat{X}_j(\omega_s)\right]^2 = \sum_{j=1}^{n}\left[\sum_{r=1}^{n}\frac{F_k\phi_{r,j}\phi_{r,k}}{(\omega_r^2-\omega_s^2)+i(2\xi_r\omega_r\omega_s)} - \hat{X}_j(\omega_s)\right]^2 \tag{19}$$

Design variables: $\phi_{1,1},\phi_{1,2},\cdots,\phi_{1,n},\phi_{2,1},\phi_{2,2},\cdots,\phi_{2,n},\phi_{n,1},\phi_{n,2},\cdots,\phi_{n,n}$ (20)

Constraints: (21)

$$[\Phi]^T[M][\Phi] = \mathrm{diag}[I] \tag{22}$$

$$[\Phi]^T[K][\Phi] \le \mathrm{diag}[\omega_r^2] \tag{23}$$

It is noted that the predicted ODS $X_j(\omega_s)$ can be calculated from Equation (18), while the measured ODS components $\hat{X}_j(\omega_s)$ can be measured in prior. One can observe that the objective function is functions of system modal parameters, including natural frequencies, damping ratios, and mode shape vectors, as well as the amplitude and excitation frequency of the harmonic force. From the ODS measurement, the system natural frequencies can be easily identified. The exciation frequency (ω_s) is also known. All of the mode shape components, i.e. design variables, can be predicted after resolving the optimization problem.

NUMERICAL RESULTS AND DISCUSSIONS

A. Three DOF system example

Consider a three DOFs system as shown in Figure 2 [13]. The equations of motion can derived as follows:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\end{Bmatrix} + \begin{bmatrix} c_1+c_4+c_6 & -c_4 & -c_6 \\ -c_4 & c_2+c_4+c_5 & -c_5 \\ -c_6 & -c_5 & c_3+c_5+c_6 \end{bmatrix}\begin{Bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{Bmatrix}$$
$$+\begin{bmatrix} k_1+k_4+k_6 & -k_4 & -k_6 \\ -k_4 & k_2+k_4+k_5 & -k_5 \\ -k_6 & -k_5 & k_3+k_5+k_6 \end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix} = \begin{Bmatrix}f_1\\ f_2\\ f_3\end{Bmatrix} \tag{24}$$

let $m_1 = 0.5\,\mathrm{kg}$, $m_2 = 1$ kg, $m_3 = 1.5$ kg, $k_i = 1000\mathrm{N/m}$, $i = 1,2,...,6$, $c_i = 10\mathrm{N\text{-}s/m}$, $i = 1,2,...,6$, for $\alpha = 0, \beta = 0.01$. By performing modal analysis, the system natural frequencies and its corresponding mode shape can be determined as follows:

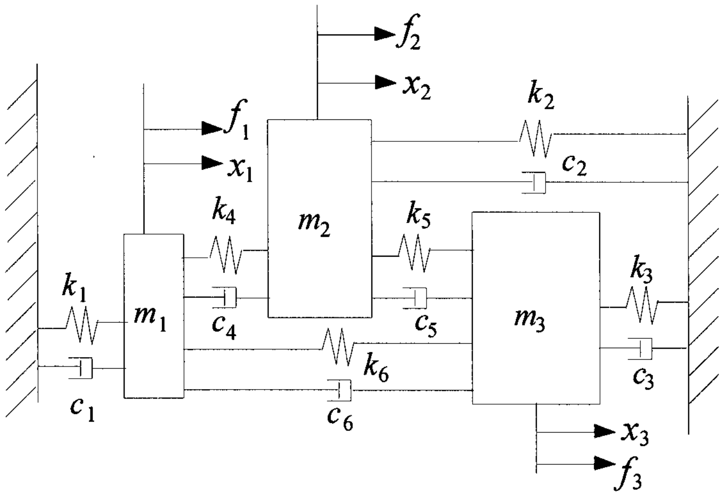

Figure 2. Example of three DOF system [13]

$$\mathrm{diag}\left[\omega_r^2\right]=\begin{bmatrix}30.82^2 & 0 & 0\\ 0 & 57.89^2 & 0\\ 0 & 0 & 81.84^2\end{bmatrix}=\begin{bmatrix}950.3 & 0 & 0\\ 0 & 3351.8 & 0\\ 0 & 0 & 6697.9\end{bmatrix} \tag{25}$$

$$[\Phi]=\left[\{\phi\}_1\,\{\phi\}_2\,\{\phi\}_3\right]=\begin{bmatrix}0.4639 & -0.2181 & 1.3181\\ 0.5361 & -0.7819 & -0.3181\\ 0.6351 & 0.4932 & -0.1419\end{bmatrix} \tag{26}$$

Assuming that F_k =1 N, k=3, ω_s=30 rad/sec, and the measured ODS $\{\hat{X}(\omega_s)\}$ is replaced by the simulated ODS, the optimization problem can be formulated and solved to predict the mode shapes. The error percentage matrix for the predicted modal matrix can be defined and obtained as follows:

$$[\varepsilon]=\frac{[\hat{\Phi}]-[\Phi]}{[\Phi]}\times 100\%=\begin{bmatrix}0.0024 & -0.0089 & -0.0001\\ 0.0014 & -0.0004 & -0.0015\\ -0.0011 & 0.0013 & 0.0065\end{bmatrix} \tag{27}$$

B. Four DOF system example

Consider another four DOF system [14], the equations of motion is as follows:

$$\begin{bmatrix}m_1 & 0 & 0 & 0\\ 0 & m_2 & 0 & 0\\ 0 & 0 & m_3 & 0\\ 0 & 0 & 0 & m_4\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\\ \ddot{x}_4\end{Bmatrix}+\begin{bmatrix}k_1+k_2 & -k_2 & 0 & 0\\ -k_2 & k_2+k_3 & -k_3 & 0\\ 0 & -k_3 & k_3+k_4 & -k_4\\ 0 & 0 & -k_4 & k_4\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{Bmatrix}=\begin{Bmatrix}f_1\\ f_2\\ f_3\\ f_4\end{Bmatrix} \tag{28}$$

Let $m_1=m_2=m_3=m_4=4000$ kg, and $k_1=k_2=k_3=k_4=5000$N/m. By performing modal analysis, the system natural frequencies and its corresponding mode shape can be determined as follows:

$$\mathrm{diag}[\omega_r^2] = \begin{bmatrix} 0.3883^2 & 0 & 0 & 0 \\ 0 & 1.1180^2 & 0 & 0 \\ 0 & 0 & 1.7129^2 & 0 \\ 0 & 0 & 0 & 2.1012^2 \end{bmatrix} = \begin{bmatrix} 0.1508 & 0 & 0 & 0 \\ 0 & 1.2500 & 0 & 0 \\ 0 & 0 & 2.9341 & 0 \\ 0 & 0 & 0 & 4.4151 \end{bmatrix} \tag{29}$$

$$[\Phi] = [\{\phi\}_1 \{\phi\}_2 \{\phi\}_3 \{\phi\}_4] = \begin{bmatrix} 0.0036 & 0.0091 & 0.0104 & 0.0068 \\ 0.0068 & 0.0091 & -0.0036 & -0.0104 \\ 0.0091 & 0.0000 & -0.0091 & 0.0091 \\ 0.0104 & -0.0091 & 0.0068 & -0.0036 \end{bmatrix} \tag{30}$$

Assuming that F_k=1 N, k=4, ω_s=0.6 rad/sec, and the measured ODS $\{\hat{X}(\omega_s)\}$ is replaced by the simulated ODS, the optimization problem can be formulated and solved to predict the mode shapes. The error percentage matrix for the predicted modal matrix can be obtained as follows:

$$[\varepsilon] = \frac{[\hat{\Phi}] - [\Phi]}{[\Phi]} \times 100\% = \begin{bmatrix} -0.0376 & 0.0004 & -0.0085 & 0.0299 \\ 0.0306 & 0.0125 & 0.0246 & -0.0257 \\ -0.0200 & 0.0002 & -0.0000 & 0.0200 \\ 0.0070 & -0.0129 & 0.0131 & -0.0212 \end{bmatrix} \tag{31}$$

This section analytically presents two case studies to demonstrate the developed algorithm in predicting the mode shapes from ODS for harmonically excited structures. The observations from the numerical simulation are discussed as follows:

1. The main idea of this work is to predict the mode shapes from one ODS data. The key point for successful prediction is the constraints of optimization problem as shown in Equations (22) and (23) that are the orthonormality of modal matrix. The orthonormality of modal matrix would guarantee the convergence of optimum solution. However, this becomes the drawback of the algorithm that requires the system mass and stiffness matrices. The future work in revising the algorithm for without the knowledge of system matrices will be more beneficial for practical application.
2. The other drawback of the proposed method is that the force amplitude should be known. Additional design variable for the force amplitude can be identified and implemented into the optimization problem. The concern, therefore, becomes the force prediction problem and is also under investigation.
3. The proposed algorithm has been tested for different initial guesses and different excitation frequencies as well as for different force locations. Results show that the algorithm is robust. Even a very bad initial guess of mode shapes can quickly converge to satisfactory optimum solutions. For either on- or off-resonance harmonic excitation case, the prediction of mode shapes is also good. The error percentage matrices shown in Equations (27) and (31) are typical and small.

CONCLUSIONS

The need in obtaining structural modal information is important for many engineering applications. The conventional modal testing is the common means to experimentally determine structural modal properties, such as natural frequencies, mode shapes and damping ratios. The major limitation of conventional modal testing is that the tested structure must be static and the controlled force input should be properly applied. Besides, in order to extract mode shapes a set of FRFs must be measured. This paper provides a method to determine the mode shape vector from single ODS information for the MDOF systems. With the prior knowledge of natural frequencies and input harmonic force amplitude and excitation frequency, single measurement of ODS is enough to determine the mode shape vector. Numerical examples for three and four DOF systems are demonstrated to prove the algorithm, respectively. Experimental verification is under investigation. The algorithm can also be extended to continuous structure as well. The work will impact the modal testing technique for the structure not necessary in static and for only one measurement of ODS required.

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