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DEVELOPMENT AND VALIDATION OF STRUCTURAL MODAL ANALYSIS BY FREE VIBRATION RESPONSE ONLY: FREQUENCY DOMAIN METHOD

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This paper aims to develop the output-only modal analysis technique to overcome some disadvantages of conventional experimental modal analysis (EMA), such as the requirement of controllable excitation sources, i.e. the impact hammer or shaker, and the limitation of test structure in static condition. This work assumes the displacement or acceleration response of the structural system due to initial conditions in free vibration can be measured. The displacement, velocity and acceleration response matrices in time and frequency domains can then be calculated by numerical methods, respectively. The theoretical approach for modal analysis by free vibration response only (MAFVRO) of MDOF system is derived. In particular, the eigenvalue problem can be formulated from the response matrices and solved for its eigenvalues and eigenvectors that can be physically interpreted as the structural modal parameters. The developed MAFVRO algorithm is applied to the MDOF systems and a cantilever beam structure to obtain structural modal parameters and well validated to show the feasibility of the algorithm. This paper proposes a brand new output-only modal analysis approach applicable to arbitrary engineering structures and enhances to promote the modal testing technique useful for industrial applications.

1. Introduction

Experimental modal analysis (EMA) or modal testing [1, 2] is the well known technique. In conventional EMA, the test structure is usually assumed in static. The excitation should be controllable and measurable. The impact hammer is frequently used as the actuator and the accelerometer as the sensor via a FFT analyzer to obtain the system frequency response functions (FRFs) between the output response and the excitation input. Also, the modal parameter extraction method or curve fitting technique should be applied to extract the modal parameters from a set of FRFs to determine structural modal parameters, including natural frequencies, mode shapes, and modal damping ratios.

Operational modal analysis (OMA) [3-5], output-only modal analysis (OOMA) [6-8] or natural input modal analysis (NIMA) [9] is of interest to overcome the disadvantage of EMA. Wang and Cheng [10] proposed an algorithm of modal analysis from free vibration response only (MAFVRO). Their formulation is limited to the proportional viscous damping base on normal mode analysis. The natural frequencies and mode shape vectors for MDOF systems can be successfully obtained. This work extends the MAFVRO [10] to general or non-proportional viscous damping cases. The complex mode analysis is adopted and thus the natural frequencies, mode shapes and modal damping ratios can be determined, simultaneously. Also, the time domain method for MAFVRO [10] is extended to the frequency domain method. Both methods are studied and compared for their effectiveness in modal analysis.

This work derives the algorithm of MAFVRO for both the proportional and non-proportional viscous damping models. Section 2 shows the detail development of MAFVRO algorithm for the time domain method. Section 3 lays out the approach of MAFVRO in the frequency domain method. Section 4 demonstrates the case studies for employing MAFVRO to simulate modal analysis on the MDOF system and a beam structure.

2. MAFVRO formulation: time domain method

Consider a MDOF vibration system with viscous damping. The general form of equation of motion can be expressed as follows:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \,. \tag{1}$$

The initial conditions are

$$\mathbf{x}(0) = \mathbf{x_0}\,, \tag{2}$$

$$\dot{\mathbf{x}}(0) = \mathbf{v_0}\,. \tag{3}$$

2.1 Proportional viscous damping model

For the proportional viscous damping, the following relation holds:

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}\,. \tag{4}$$

For normal mode analysis, let

$$\mathbf{x}\text{=}\mathbf{X}\,e^{i\omega t}\,\mathbf{,} \tag{5}$$

By the substitution of Eq. (5) into Eq. (1) and the assumptions of f=0 and C=0, the generalized eigenvalues problem can be formulated:

$$\mathbf{KX}\text{=}\,\omega^2\,\mathbf{MX}. \tag{6}$$

or

$$\mathbf{M}^{-1}\mathbf{KX} = \omega^2\mathbf{X}\,. \tag{7}$$

By solving the above equation, n-pairs of eigenvalues ω_r^2 and eigenvector $\mathbf{X_r}$ can be obtained. Physically, $\omega_r = 2\pi f_r$ is the r-th natural frequency, and $\mathbf{X_r} = \boldsymbol{\phi}_\mathbf{r}$ is its corresponding mode shape vector.

The following derivation is partly adopted from Wang and Cheng [10]. This work is to determine modal parameters, including natural frequencies ω_r and mode shape $\boldsymbol{\phi}_\mathbf{r}$, from the free vibration response, i.e, $\mathbf{f(t)} = \mathbf{0}$. For the proportional viscous damping model without the prescribed force, the system equation becomes

$$\mathbf{M}\ddot{\mathbf{x}} + (\alpha\mathbf{M} + \beta\mathbf{K})\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}\,. \tag{8}$$

Rearrange the above equation

$$\mathbf{M}(\ddot{\mathbf{x}} + \alpha\dot{\mathbf{x}}) = -\mathbf{K}(\mathbf{x} + \beta\dot{\mathbf{x}})\,. \tag{9}$$

Then

$$\mathbf{M}^{-1}\mathbf{K} = -(\ddot{\mathbf{x}} + \alpha\dot{\mathbf{x}})(\mathbf{x} + \beta\dot{\mathbf{x}})^{-1}\,. \tag{10}$$

By comparing Equations (10) and (7), one can conclude that if the system response $\mathbf{x}$, $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$ are known, $\mathbf{M}^{-1}\mathbf{K}$ can be formulated and used to solve for the eigenvalues and eigenvectors, i.e. the normal modes of the system. Consider the system displacement response matrix as follows:

$$\mathbf{X}(t)=[X]_{N_k\times n}=\begin{bmatrix} x_{1,k} & x_{2,k} & \cdots & x_{n,k} \\ x_{1,k+1} & x_{2,k+1} & \cdots & x_{n,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,k+N_k-1} & x_{2,k+N_k-1} & \cdots & x_{n,k+N_k-1} \end{bmatrix}=\left[\begin{Bmatrix} x_{1,k} \\ x_{2,k} \\ \vdots \\ x_{n,k}\end{Bmatrix}\begin{Bmatrix} x_{1,k+1} \\ x_{2,k+1} \\ \vdots \\ x_{n,k+1}\end{Bmatrix}\cdots\begin{Bmatrix} x_{1,k+N_k-1} \\ x_{2,k+N_k-1} \\ \vdots \\ x_{n,k+N_k-1}\end{Bmatrix}\right]^{\mathrm{T}}, \tag{11}$$

$$=\left[\{x\}_k\,\{x\}_{k+1}\cdots\{x\}_{k+N_k-1}\right]^{\mathrm{T}}$$

where $x_{r,k}=x_r(t_k)$ denotes the displacement of the r-th DOF at time t_k. Similarly, the system velocity and acceleration response matrix can be defined

$$\dot{\mathbf{X}}=\left[\dot{X}\right]=\left[\{\dot{x}\}_k\,\{\dot{x}\}_{k+1}\cdots\{\dot{x}\}_{k+N_k-1}\right]^{\mathrm{T}}, \tag{12}$$

$$\ddot{\mathbf{X}}=\left[\ddot{X}\right]=\left[\{\ddot{x}\}_k\,\{\ddot{x}\}_{k+1}\cdots\{\ddot{x}\}_{k+N_k-1}\right]^{\mathrm{T}}. \tag{13}$$

Eq. (10) can then be rewritten as follows:

$$\mathbf{M}^{-1}\mathbf{K}=-(\ddot{\mathbf{X}}^{\mathbf{T}}+\alpha\dot{\mathbf{X}}^{\mathbf{T}})(\beta\dot{\mathbf{X}}^{\mathbf{T}}+\mathbf{X}^{\mathbf{T}})^{-1}. \tag{14}$$

2.2 Non-proportional viscous damping model

For the general or non-proportional viscous damping, the following equilibrium equation is invoked:

$$\mathbf{M}\dot{\mathbf{x}}-\mathbf{M}\dot{\mathbf{x}}=0. \tag{15}$$

By combining Eqs. (1) and (15), the system equation can rewritten as follows:

$$\mathbf{A}\dot{\mathbf{y}}+\mathbf{B}\mathbf{y}=\mathbf{P} \tag{16}$$

where

$$\mathbf{A}=\begin{bmatrix}\mathbf{0} & \mathbf{M}\\ \mathbf{M} & \mathbf{C}\end{bmatrix}, \mathbf{B}=\begin{bmatrix}-\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}\end{bmatrix},\ \mathbf{y}=\begin{Bmatrix}\dot{\mathbf{x}}\\ \mathbf{x}\end{Bmatrix}, \mathbf{P}=\begin{Bmatrix}\mathbf{0}\\ \mathbf{f}\end{Bmatrix}. \tag{17}$$

Let

$$\mathbf{y}=\mathbf{Y}e^{i\lambda t}. \tag{18}$$

By the substitution of above Eq. into Eq. (16) and the assumption of zero external force vectors $\mathbf{f}$ =0, i.e. $\mathbf{P}$ =0, the eigenvalue problem can be formulated as follows:

$$\mathbf{BY}=-\lambda\mathbf{AY} \tag{19}$$

or

$$(-\mathbf{A}^{-1}\mathbf{B})\mathbf{Y}=\lambda\mathbf{Y}. \tag{20}$$

By solving the above equation, 2n pairs of complex conjugate eigenvalues and their corresponding eigenvectors can be obtained:

$$\begin{cases}\lambda_r\to\mathbf{Y}_r\\ \lambda_r^*\to\mathbf{Y}_r^*\end{cases},\ r=1,2,...,n \tag{21}$$

where

$$\begin{matrix}\lambda_r\\ \lambda_r^*\end{matrix}=R_e\pm iI_m=-\bar{\zeta}_r\bar{\omega}_r\pm i\,\bar{\omega}_r\sqrt{1-\bar{\zeta}_r^{\,2}}\,. \tag{22}$$

The equivalent natural frequency and modal damping ratio can be determined:

$$\bar{\omega}_r=\sqrt{R_e^2+I_m^2} \tag{23}$$

$$\bar{\zeta}_r = \frac{-R_e}{\sqrt{R_e^2 + I_m^2}} \tag{24}$$

and

$$\mathbf{Y}_r = \begin{Bmatrix} \dot{\overline{\mathbf{X}}}_r \\ \overline{\mathbf{X}}_r \end{Bmatrix},\ \mathbf{Y}_r^* = \begin{Bmatrix} \dot{\overline{\mathbf{X}}}_r^* \\ \overline{\mathbf{X}}_r^* \end{Bmatrix} \tag{25}$$

where $\bar{\omega}_r = 2\pi \bar{f}_r$, $\bar{\zeta}_r$ and $\bar{\boldsymbol{\phi}}_\mathbf{r} = \overline{\mathbf{X}}_\mathbf{r}$ are the r-th natural frequency, modal damping ratio and displacement mode shape vector, respectively. The bar symbol is to denote the solutions from the complex mode analysis, in particular for the non-proportional viscous damping.

Similar to the derivation of the proportional viscous damping for MAFVRO, the system equation in Eq. (16) without the prescribed force, i.e. $\mathbf{P}=0$, is as follows:

$$\mathbf{A}\dot{\mathbf{y}} + \mathbf{B}\mathbf{y} = \mathbf{0}. \tag{26}$$

One can obtain

$$-\mathbf{A}^{-1}\mathbf{B} = \dot{\mathbf{y}}\mathbf{y}^{-1} = \begin{Bmatrix} \ddot{\mathbf{x}} \\ \dot{\mathbf{x}} \end{Bmatrix} \begin{Bmatrix} \dot{\mathbf{x}} \\ \mathbf{x} \end{Bmatrix}^{-1}. \tag{27}$$

From the definition of the system displacement, velocity and acceleration response matrices as shown in Eqs. (11)-(13), the above equation become

$$-\mathbf{A}^{-1}\mathbf{B} = \begin{Bmatrix} \ddot{\mathbf{X}}^\mathrm{T} \\ \dot{\mathbf{X}}^\mathrm{T} \end{Bmatrix} \begin{Bmatrix} \dot{\mathbf{X}}^\mathrm{T} \\ \mathbf{X}^\mathrm{T} \end{Bmatrix}^{-1}. \tag{28}$$

By comparing Eqs. (20) and (28), one can see that if the system response matrices are known, $-\mathbf{A}^{-1}\mathbf{B}$ can be formulated and used to solve the eigenvalues and eigenvectors. Therefore, the system modal parameters as shown in Eqs. (23)-(25) can be obtained. This approach is the main idea of MAFVRO for the non-proportional viscous damping.

If the displacement sensor is used to measure the system displacement response, $x_r(t_k) = x_{r,k}$, as illustrated in Fig. 1, the velocity and acceleration can be determined by finite difference method. Wang and Cheng [10] showed detail matrix operation among response matrices. If the accelerometer is used, the acceleration at each DOF $\ddot{x}_r(t_k) = \ddot{x}_{r,k}$ can be measured. The numerical formula can be adopted to evaluate the velocity and displacement, respectively. Therefore, Eqs. (14) and (28)

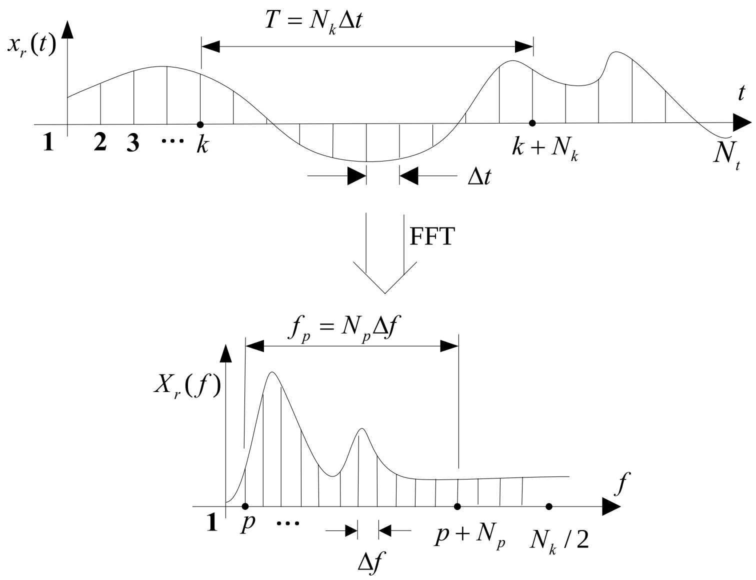


Figure 1. Diagram for the time and frequency domain response.

can be obtained from the free vibration response and solved for modal parameters, ω_r and $\boldsymbol{\phi}_{\mathbf{r}}$.

3. MAFVRO formulation: frequency domain method

Section 2 shows the MAFVRO algorithm in the time domain. This section will illustrate the formulation of MAFVRO in the frequency domain. From Eqs. (11)-(13), the time domain response matrices can be measured or numerically determined. The basic idea is to determine the Fourier spectrum of the system responses. For example, the *r*-th DOF time domain data $x_r(t)$ can be theoretically performed Fourier transform as follows:

$$X_r(f) = \boldsymbol{F}[x_r(t)] \tag{29}$$

where $\boldsymbol{F}$ denotes the Fourier transform operator. $X_r(f)$ denotes the Fourier spectrum of $x_r(t)$. Fig. 1 shows the diagram of discrete time and frequency domain response. Similar to Eq. (11), the system displacement Fourier spectra response matrix can be written as follows:

$$\mathbf{X}(f) = [X]_{N_p \times n} = \begin{bmatrix} X_{1,p} & X_{2,p} & \cdots & X_{n,p} \\ X_{1,p+1} & X_{2,p+1} & \cdots & X_{n,p+1} \\ \vdots & \vdots & \ddots & \vdots \\ X_{1,p+N_p-1} & X_{2,p+N_p-1} & \cdots & X_{n,p+N_p-1} \end{bmatrix} = \begin{bmatrix} \begin{Bmatrix} X_{1,p} \\ X_{2,p} \\ \vdots \\ X_{n,p} \end{Bmatrix} \begin{Bmatrix} X_{1,p+1} \\ X_{2,p+1} \\ \vdots \\ X_{n,p+1} \end{Bmatrix} \cdots \begin{Bmatrix} X_{1,p+N_p-1} \\ X_{2,p+N_p-1} \\ \vdots \\ X_{n,p+N_p-1} \end{Bmatrix} \end{bmatrix}^{\mathrm{T}} \tag{30}$$

$$= \left[\{X\}_p \{X\}_{p+1} \cdots \{X\}_{p+N_p-1} \right]^{\mathrm{T}}$$

where *p* and N_p are the start number and end number of the frequency domain data used to formulate the response matrix. Similarly, the velocity and acceleration Fourier spectra can also be obtained. Therefore, the Fourier spectra response matrices for displacement, velocity and acceleration can be easily substituted into Eqs. (14) and (28) to formulate the eigenvalue problems for both the proportional and non-proportional viscous damping models.

From Fig. 1, assuming that N_k time data are used and starting from *k* point, the corresponding Fourier spectrum $X_r(f)$ can be revealed in Fig. 1. The relations of those variables in Fig. 1 are summarized as follows:

$$\Delta f = \frac{1}{T} = \frac{1}{N_k \Delta t}, \tag{31}$$

$$f_{\mathrm{nyq}} = \frac{f_s}{2} = \frac{N_k \Delta f}{2}. \tag{32}$$

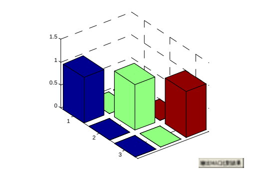
where Δf is the frequency resolution. T is the time period for taking numerical Fourier transform. f_s and f_{nyq} are the sampling frequency and Nyquist frequency, respectively. For formulating the MAFVRO algorithm in the frequency domain method, both Eqs. (14) and (28) for the proportional and non-proportional viscous damping models can be simply replaced by the frequency domain data. To implement the algorithm, let those frequency data from p to $p + N_p$ to be chosen such that the frequency range is $f_p = N_p \Delta f$.

4. Results and Discussions

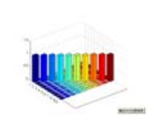
This section will employ the developed MAFVRO algorithms in both the time and frequency domain methods for both the proportional and non-proportional viscous damping models to obtain the structural modal parameters via simulation data.

Table 1. Modal parameters prediction results for MDOF systems.

(a) 3-DOF system

Mode		1	2	3	MAC plot
$\bar{f}_r$	TMA	70.8306	198.4630	286.7872	
	MAFVRO	70.8305	198.4507	286.7103	
	Error(%)	-0.0001	-0.0062	-0.0268	
$\bar{\zeta}_r$	TMA	0.0242	0.0436	0.0194	
	MAFVRO	0.0242	0.0435	0.0194	
	Error(%)	-0.0004	-0.0247	-0.1068	

(b) 10-DOF system

Mode		1	2	3	4	5	6	7	8	9	10	MAC plot
$\bar{f}_r$	TMA(Hz)	23.7873	70.8306	116.2917	159.1550	198.4630	233.3377	262.9999	286.7873	304.1682	314.7546	
	MAFVRO(Hz)	23.7873	70.8305	116.2908	159.1509	198.4507	233.3101	262.9500	286.7103	304.0651	314.6324	
	Error(%)	-0.0000	-0.0001	-0.0007	-0.0026	-0.0062	-0.0118	-0.0190	-0.0268	-0.0339	-0.0388	
$\bar{\zeta}_r$	TMA(%)	0.0028	0.0081	0.0121	0.0143	0.0145	0.0129	0.0100	0.0065	0.0032	0.0008	
	MAFVRO(%)	0.0028	0.0081	0.0121	0.0143	0.0145	0.0129	0.0100	0.0065	0.0032	0.0008	
	Error(%)	0.0001	-0.0004	-0.0029	-0.0102	-0.0247	-0.0470	-0.0757	-0.1068	-0.1348	-0.1545	

Note: k=5, N_k=100, p=1, N_p=20, Δf=60 Hz, f_p=1200 Hz, f_s=6000Hz, N_t=1000, c_1=1 $(\mathrm{N \cdot s/m})$, c_2=c_3=0 $(\mathrm{N \cdot s/m})$, SNR=0 (%)

The main difference for the proportional and non-proportional viscous damping models for MAFVRO algorithms is that the proportional model is truly the normal mode analysis, while the non-proportional model is the complex mode analysis. Tables 1(a) and 1(b) shows the modal parameters prediction results for a 3-DOF and 10-DOF systems, respectively, via the non-proportional viscous damping model for the frequency domain method. The predicted errors for natural frequencies and damping ratios are quite small, and the modal assurance criterion (MAC) matrix for the comparison of theoretical and MAFVRO predicted mode shape vectors reveals a unity matrix. Results show the MAFVRO can successfully obtain the modal parameters correctly. The feasibility of MAFVRO via the frequency domain method for the non-proportional viscous damping model is demonstrated.

The time domain method of MAFVRO via the proportional viscous damping model is previously demonstrated [10]. This work shows the merit of the frequency domain method of MAFVRO as well as the non-proportional viscous damping models. Tables 2 and 3, respectively, show the modal parameters prediction results for both the proportional and non-proportional models via the time and frequency domain methods, respectively. It is noted that the signal noise ratio (SNR) is assumed as SNR=15% and 18% as noted in the tables. The discussions are as follows:

1. From Table 2 for the proportional model, the time domain method results in up to 20% of errors for natural frequency predictions, while the frequency domain method largely improve the prediction errors within 2%.
2. For the non-proportional model shown in Table 3, the time domain method reveals high prediction errors for natural frequencies. On the contrary, the frequency domain method results in the maximum error within 2% for SNR=18%. The frequency domain method can appropriately accommodate the high SNR conditions.
3. For damping ratio prediction, the proportional model is not supported. The non-proportional model shows the reasonable range of prediction. The errors of predicted damping ratios are mainly due to the high level of SNR.
4. The mode shape predictions for both models are satisfactory well. It should be noted that the predicted mode shape vectors are real for the proportional model and complex for the non-proportional model.

By applying the developed MAFVRO algorithms to a beam structure, the free vibration response of the beam should be measured in some grid points over the beams. The thin beam model is assumed to obtain the free vibration response for simulation purpose. Assuming there are 14 points along the beam length to measure the beam response, i.e. m=14. Therefore, the number of DOFs becomes n=m.

Table 2. Comparison of modal parameters prediction results for proportional viscous damping of MAFVRO by displacement sensors.

(a) Time domain method

MAFVRO method	System model	Theoretical natural freq. (Hz)	Predicted natural freq. (Hz)	Error of natural freq. (%)	Theoretical damping ratio (%)	Predicted damping ratio (%)	Error of damping ratio (%)
proportional viscous damping	proportional viscous damping	70.8306	74.6321	**5.3670**	2.2252	-	-
		198.4630	228.2450	**15.0063**	6.2349	-	-
		286.7873	357.7089	**24.7297**	9.0097	-	-
	non-proportional viscous damping	70.8306	70.9244	0.1323	0.0242	-	-
		208.7925	208.7925	**5.2048**	0.0436	-	-
		286.7872	306.6965	**6.9422**	0.0194	-	-

Note: k =5, N_k =100, f_s =2500Hz, N_t =1000, α =0.0001, β =0.0001, SNR=15 (%)

(b) Frequency domain method

MAFVRO method	System model	Theoretical natural freq. (Hz)	Predicted natural freq. (Hz)	Error of natural freq. (%)	Theoretical damping ratio (%)	Predicted damping ratio (%)	Error of damping ratio (%)
proportional viscous damping	proportional viscous damping	70.8306	72.2006	1.9342	2.2252	-	-
		198.4630	202.9156	1.2436	6.2349	-	-
		286.7873	288.5167	0.6030	9.0097	-	-
	non-proportional viscous damping	70.8306	69.4525	-1.9457	0.0242	-	-
		198.4630	198.3673	-0.0482	0.0436	-	-
		286.7872	282.4861	-1.4998	0.0194	-	-

Note: k =5, N_k =100, p =1, N_p =20, Δf =25 Hz, f_p =500 Hz, f_s =2500Hz, N_t =1000, α =0.0001, β =0.0001, SNR=15 (%)

Table 3. Comparison of modal parameters prediction results for non-proportional viscous damping of MAFVRO by displacement sensors.

(a) Time domain method

MAFVRO method	System model	Theoretical natural freq. (Hz)	Predicted natural freq. (Hz)	Error of natural freq. (%)	Theoretical damping ratio (%)	Predicted damping ratio (%)	Error of damping ratio (%)
non-proportional viscous damping	proportional viscous damping	70.8306	71.3428	0.7231	2.2252	3.4737	**56.1059**
		198.4630	229.0680	**15.4210**	6.2349	6.8150	**9.3037**
		286.7873	370.7638	**29.2818**	9.0097	1.6238	**-81.9770**
	non-proportional viscous damping	70.8306	73.6255	**3.9459**	0.0242	1.8021	**7355.4**
		198.4630	210.4340	**6.0319**	0.0436	**-0.1487**	**-441.4**
		286.7872	320.3666	**11.7088**	0.0194	0.1528	**688.1**

Note: k =5, N_k =100, f_s =2500 Hz, N_t =1000, c_1=1 $(\mathrm{N\cdot s/m})$, c_2=c_3=0 $(\mathrm{N\cdot s/m})$, SNR=18 (%)

(b) Frequency domain method

MAFVRO method	System model	Theoretical natural freq. (Hz)	Predicted natural freq. (Hz)	Error of natural freq. (%)	Theoretical damping ratio (%)	Predicted damping ratio (%)	Error of damping ratio (%)
non-proportional viscous damping	proportional viscous damping	70.8306	70.9339	0.1458	2.2252	0.7784	**-65.0186**
		198.4630	202.0860	1.8256	6.2349	4.3767	**-29.8040**
		286.7873	281.2662	-1.9252	9.0097	7.0120	**-22.1725**
	non-proportional viscous damping	70.8306	70.1896	-0.9050	0.0242	0.2426	**903.5082**
		198.4630	199.5359	0.5406	0.0436	**-0.2599**	**-696.6929**
		286.7872	287.1032	0.1102	0.0194	0.1776	**815.9760**

Note: k =5, N_k =100, p =1, N_p =20, Δf =25Hz, f_p =500Hz, f_s =2500Hz, N_t =1000, c_1=1 $(\mathrm{N\cdot s/m})$, c_2=c_3=0 $(\mathrm{N\cdot s/m})$, SNR=18 (%)

By considering a steel cantilever beam, both the proportional and non-proportional models are adopted to demonstrate the feasibility of MAFVRO algorithms for the continuous structure applications. Table 4(a) shows the prediction results for the proportional model, either the natural frequencies or mode shape predictions are very well. The maximum errors for natural frequencies are mostly within 1%, and the MAC values between the predicted and theoretical mode shapes are nearly close to 1.For the non-proportional model as shown in Table 4(b), both natural frequencies and mode shapes also reveals very good predictions, although the damping ratios may reveal high errors or even unreasonable negative values. In practical EMA, the most difficult part is to identify the structural mode shapes. The presented MAFVRO algorithm does provide an effective way to obtain the modal parameters from only the free vibration response.

Table 4. Modal parameter prediction for the cantilever beam.

(a) Proportional model

mode	Natural frequency (Hz)			MAC	Mode shape
	TMA	MAFVRO	Err(%)		
1	16.328	16.367	0.23924	1	
2	102.33	102.32	-0.004671	1	
3	286.52	290.07	**1.2392**	0.99841	
4	561.47	561.79	0.056622	1	
5	928.15	927.89	-0.027653	1	
6	1386.5	1386.1	-0.030038	1	
7	1936.5	1935.9	-0.029011	1	
8	2578.2	2577.4	-0.030727	1	
9	3311.5	3310.5	-0.031196	1	
10	4136.6	4135.1	-0.034916	1	
11	5053.3	5050.6	-0.052775	1	
12	6061.6	6057.5	-0.0679	1	
13	7161.7	7173.7	0.16778	0.99997	
14	8353.4	8339.4	-0.16764	0.99987	

(b) Non-proportional model

mode	Natural freq. (Hz)			Damping ratio (%)			MAC	Mode shape
	TMA	MAFVRO	Err(%)	TMA	MAFVRO	Err		
1	16.328	16.432	0.63571	0.487	**-0.16015**	**-132.89**	0.99987	
2	102.33	102.2	-0.12741	0.0778	0.11052	42.053	1	
3	286.52	291.33	**1.6794**	0.0278	**-2.8475**	**-10343**	0.99649	
4	561.47	561.37	-0.017571	0.0142	0.0039488	-72.191	1	
5	928.15	927.83	-0.034707	0.00857	0.0061062	-28.749	1	
6	1386.5	1386.1	-0.027238	0.00574	0.0062791	9.3928	1	
7	1936.5	1936.1	-0.022262	0.00411	0.0028877	-29.739	1	
8	2578.2	2577.4	-0.029764	0.00309	0.0028004	-9.3736	1	
9	3311.5	3310.4	-0.035017	0.0024	0.0034949	45.621	1	
10	4136.6	4135.1	-0.035103	0.00192	**-0.00020228**	**-110.54**	0.99999	
11	5053.3	5052	-0.024936	0.00157	**-0.0072817**	**-563.81**	1	
12	6061.6	6057.5	-0.067841	0.00131	0.0013813	5.4464	1	
13	7161.7	7178.1	0.22943	0.00111	0.0060248	**442.78**	0.99996	
14	8353.4	8339.4	-0.16709	0.000953	0.0011956	25.455	0.99994	

5. Conclusions

This work mainly extends the MAFVRO algorithm from the time domain method [10] to the frequency domain method. Both the proportional and non-proportional viscous damping models are developed for the MAFVRO algorithms. The proportional model is truly the normal mode analysis, while the non-proportional model is the complex mode analysis which is more appropriate for the need of practical structural modal analysis. In this paper, the applications of MAFVRO algorithm to the MDOF system and a beam, i.e. the continuous structure, are demonstrated for the potential and feasible use in experimental modal analysis. The modal parameters can be successfully predicted by the MAFVRO algorithm. This work enhances the modal analysis technique by using the free vibration response only.

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