Laminate Plate Theory for Spatially Distributed Induced Strain Actuators

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ABSTRACT: Classical laminated plate theory (CLPT) is applied to a laminate plate with induced strain actuators, such as piezoceramic patch, bonded to its surface or embedded within the laminate to develop an induced strain actuation theory that allows for the actuator patch to be spatially distributed. When piezoceramic patches are subjected to voltage fields, the equivalent external forces induced by piezoceramic patches can be determined upon the assumption of free constraint for the expansion or contraction of piezoceramic patches. This assumption is generally done in thermal expansion problem. Several examples, including pure bending and pure extension, are illustrated. For the case of pure bending, a comparison between the current work and that of Dimitriadis et al. (1989) is given. In addition, an orthotropic angle-ply laminate with an embedded piezoceramic patch is presented to show the coupling of bending and extension.

INTRODUCTION

DISTRIBUTED INDUCED STRAIN actuators, such as piezoceramic patches, are often chosen for noise and vibration controls, primarily because of their low cost, light weight and easy implementation. Fanson and Chen (1986) showed the feasibility of using piezoelectric materials as actuators and sensors in beam vibration controls. They introduced the concept of piezoelectric active members to replace passive structural elements for the control of large space structures (LSS). In addition, Meirovitch and Norris (1984) showed that the use of distributed controls, such as piezoelectric actuators, is more beneficial than the use of point controls. Therefore, researchers have been increasingly interested in the application of piezoelectric actuators to structural vibration and noise controls.

Bailey and Hubbard (1985) applied the distributed parameter control theory and distributed-parameter actuators to avoid the truncation of the model for the control of bending vibration in beams. Distributed actuators which can generate a number of modes simultaneously and reduce the control spillover are desirable to remedy the drawbacks of point actuators. Crawley and de Luis (1987) devel-

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oped an elastic model for one-dimensional piezoceramic patches bonded to the surface or embedded into the body of beams. They showed that distributed actuators perfectly bonded result in two equivalent concentrated moments acting at the edges of an actuator patch. Dimitriadis et al. (1989) presented a two-dimensional model for piezoceramic patches bonded to the top and bottom surfaces of a rectangular plate and showed that the resultant moments induced by the piezoceramic patches were along the four edges of piezoceramic patches under the assumption of pure spherical bending. Wang et al. (1989) employed the model developed by Dimitriadis et al. (1989) and used multiple piezoelectric actuators to control the noise radiation of simply-supported, baffled rectangular plates. They showed that appropriately positioned actuators reduce the undesired control spillover.

In addition to the use of distributed induced strain actuators in vibration or noise control, the design of distributed induced strain actuators has been encouragingly investigated. Lee (1987) applied the classical laminate plate theory to design piezoelectric laminate for bending and torsional modal controls. His experimental results showed that PVDF or PVF_2 (polyvinylidene fluoride) actuators can generate plate bending and twisting independently or simultaneously, and PVDF are suitable for active damping control of a flexible structure. Lazarus and Crawley (1989) developed the pin-force and consistent plate models for the design of induced strain actuators. Exact solutions can be found only for the unconstrained boundary conditions. They also employed the Ritz assumed mode method to solve the problems with the boundary conditions other than the unconstrained cases.

This work applied the classical laminate plate theory (CLPT) (Jones, 1975) to model the laminated plate with spatially distributed, small size induced strain actuator patches embedded at any location of the laminated plate and to determine the equivalent external loads induced by the actuator patches under the applied voltage fields. The laminate was assumed to be perfectly bonded, and Kirchhoff's assumptions were maintained. Besides, the thickness and size of actuator patches are assumed to be relatively smaller than those of each lamina, so actuator patches can be neglected for calculating the global properties of the laminate. Several case studies were presented to show that the bending and extension can be induced separately or coupled by actuators.

THEORETICAL ANALYSIS

Let us consider a rectangular laminated plate with multiple, embedded induced strain actuators, such as piezoceramic patches. Figure 1 shows the arrangement and coordinates of the actuator plate model. Under the plane stress state, the stress–strain relations for a lamina in 12-coordinate reduce to

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} \tag{1}$$

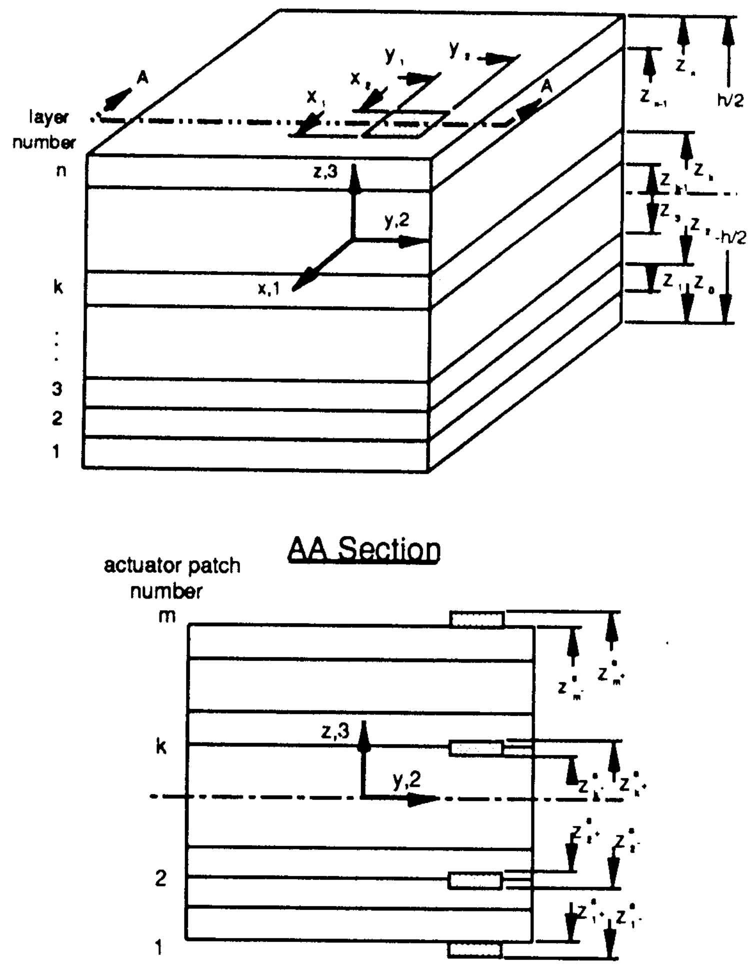


Figure 1. Arrangement and coordinate of actuator-plate model.

or in short form

$$\{\bar{\sigma}\} = [Q]\{\bar{\epsilon}\} \tag{2}$$

Then the stress-strain relations for a lamina in xy-coordinate can be shown as:

$$\{\sigma\} = [\bar{Q}]\{\epsilon\} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{3}$$

The expressions for Q_{ij} and $\overline{Q}_{ij}$ can be easily found (Jones, 1975). The total strain can be shown as the sum of the mechanical and induced actuator strains.

$$\{\epsilon\} = \{\epsilon^m\} + \{\Lambda\} \tag{4}$$

Under the Kirchhoff's assumption, the mechanical strain vector is given by

$$\{\epsilon^m\} = \{\epsilon^0\} + z\{\varkappa\} \tag{5}$$

where the midplane mechanical strains:

$$\{\epsilon^0\} = \begin{bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v^0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{bmatrix} \tag{6}$$

the midplane curvatures:

$$\{\varkappa\} = \begin{bmatrix} \varkappa_x \\ \varkappa_y \\ \varkappa_{xy} \end{bmatrix} = - \begin{bmatrix} \dfrac{\partial^2 w_0}{\partial x^2} \\ \dfrac{\partial^2 w_0}{\partial y^2} \\ 2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{bmatrix} \tag{7}$$

and the actuator strains:

$$\{\Lambda\} = \begin{bmatrix} \Lambda_x \\ \Lambda_y \\ \Lambda_{xy} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{m} [H(z - z_{k^-}^a) - H(z - z_{k^+}^a)] \dfrac{(d_{31})_k}{z_{k^+}^a - z_{k^-}^a} V_k R_k(x,y) \\ \sum_{k=1}^{m} [H(z - z_{k^-}^a) - H(z - z_{k^+}^a)] \dfrac{(d_{32})_k}{z_{k^+}^a - z_{k^-}^a} V_k R_k(x,y) \\ \sum_{k=1}^{m} [H(z - z_{k^-}^a) - H(z - z_{k^+}^a)] \dfrac{(d_{36})_k}{z_{k^+}^a - z_{k^-}^a} V_k R_k(x,y) \end{bmatrix} \tag{8}$$

where the Heaviside function, $H(z - z_0)$, is defined as follows:

$$\begin{aligned} H(z - z_0) &= 1, z \geq z_0 \\ &= 0, z < z_0 \end{aligned} \tag{9}$$

and the generalized location function is defined as:

$$\begin{aligned} R_k(x,y) &= 1, (x_1)_k \leq x \leq (x_2)_k, (y_1)_k \leq y \leq (y_2)_k \\ &= 0, \text{elsewhere} \end{aligned} \tag{10}$$

For simply application, it is assumed that each actuator patch has the same piezoelectric strain coefficient, i.e., $(d_{ij})_1 = (d_{ij})_2 = \cdots = (d_{ij})_m = d_{ij}$, and the same location on the xy-plane, i.e., $R_1 = R_2 = \cdots = R_m = R$. Let $R(x,y)$ be expressed with the Heaviside function:

$$R(x,y) = [H(x - x_1) - H(x - x_2)][H(y - y_1) - H(y - y_2)] \tag{11}$$

Also, the derivatives of generalized location function are expressed as:

$$\frac{\partial R}{\partial x} = [\delta(x - x_1) - \delta(x - x_2)][H(y - y_1) - H(y - y_2)] \tag{12}$$

$$\frac{\partial R}{\partial y} = [H(x - x_1) - H(x - x_2)][\delta(y - y_1) - \delta(y - y_2)] \tag{13}$$

$$\frac{\partial^2 R}{\partial x \partial y} = [\delta(x - x_1) - \delta(x - x_2)][\delta(y - y_1) - \delta(y - y_2)] \tag{14}$$

$$\frac{\partial^2 R}{\partial x^2} = [\delta'(x - x_1) - \delta'(x - x_2)][H(y - y_1) - H(y - y_2)] \tag{15}$$

$$\frac{\partial^2 R}{\partial y^2} = [H(x - x_1) - H(x - x_2)][\delta'(y - y_1) - \delta'(y - y_2)] \tag{16}$$

The mathematical interpretation of the Heaviside and Delta functions is illustrated in Figure 2. As $[H(x - x_1) - H(x - x_2)]$ represents a uniform distribution between x_1 and x_2, $[\delta(x - x_1) - \delta(x - x_2)]$ represents two concentrated sources at x_1 and x_2 respectively, and $[\delta'(x - x_1) - \delta'(x - x_2)]$ represents two moment sources at x_1 and x_2, respectively. So the derivatives of generalized location function can be graphically shown in Table 1. The physical meanings of the derivatives of the generalized location function will be discussed further.

Equation (3) shows the stress–strain relation for the k-th layer lamina. If Equation (3) were integrated through the thickness of the laminate, so were Equation

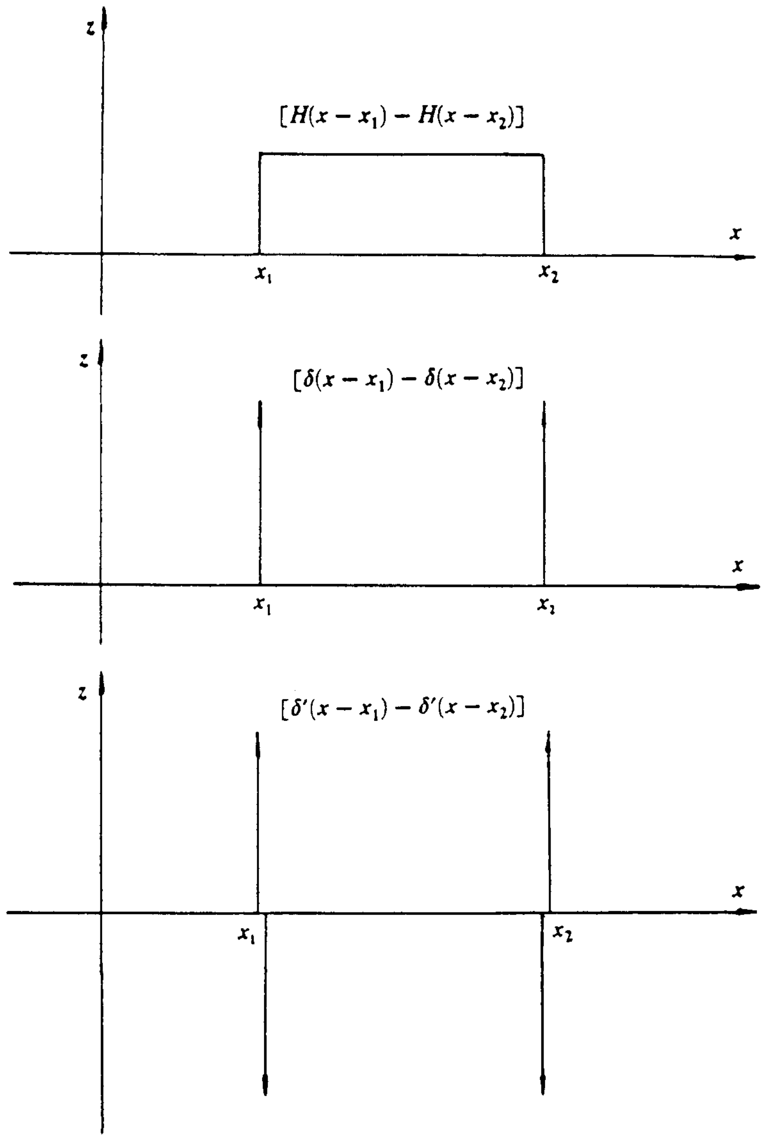
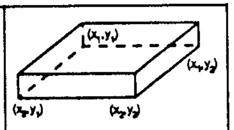
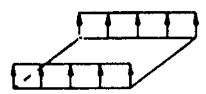
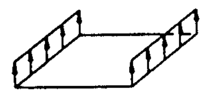
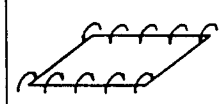
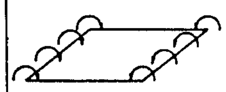
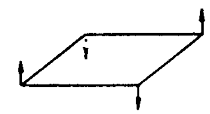


Figure 2. *Illustrations of Heaviside and Delta functions.*

Table 1. Generalized location function and its derivative.

$R(x,y) = [H(x - x_1) - H(x - x_2)][H(y - y_1) - H(y - y_2)]$	
$\frac{\partial R}{\partial x} = [\delta(x - x_1) - \delta(x - x_2)][H(y - y_1) - H(y - y_2)]$	
$\frac{\partial R}{\partial y} = [H(x - x_1) - H(x - x_2)][\delta(y - y_1) - \delta(y - y_2)]$	
$\frac{\partial^2 R}{\partial x^2} = [\delta'(x - x_1) - \delta'(x - x_2)][H(y - y_1) - H(y - y_2)]$	
$\frac{\partial^2 R}{\partial y^2} = [H(x - x_1) - H(x - x_2)][\delta'(y - y_1) - \delta'(y - y_2)]$	
$\frac{\partial^2 R}{\partial x \partial y} = [\delta(x - x_1) - \delta(x - x_2)][\delta(y - y_1) - \delta(y - y_2)]$	

(3) multiplied by z and integrated through the thickness of the laminate, then the following equations can be obtained.

$$\{N\} = [A]\{\epsilon^0\} + [B]\{\kappa\} - [E]\{d\} \quad (17)$$

and

$$\{M\} = [B]\{\epsilon^0\} + [D]\{\kappa\} - [F]\{d\} \quad (18)$$

where

$$\{N\} = \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dz \quad (19)$$

$$\{M\} = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} z\,dz \tag{20}$$

$$\{d\} = \begin{bmatrix} d_{31}R \\ d_{32}R \\ d_{36}R \end{bmatrix} \tag{21}$$

The i-th row and j-th column element of matrices $[A]$, $[B]$, $[D]$, $[E]$, and $[F]$ is as follows:

$$A_{ij} = \sum_{k=1}^{n} (\overline{Q}_{ij})_k (z_k - z_{k-1}) \tag{22}$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} (\overline{Q}_{ij})_k (z_k^2 - z_{k-1}^2) \tag{23}$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} (\overline{Q}_{ij})_k (z_k^3 - z_{k-1}^3) \tag{24}$$

$$E_{ij} = \sum_{k=1}^{m} (\overline{Q}_{ij})_k^a V_k \tag{25}$$

$$F_{ij} = \frac{1}{2} \sum_{k=1}^{m} (\overline{Q}_{ij})_k^a V_k (z_{k^+}^a + z_{k^-}^a) \tag{26}$$

The equilibrium equations are given as:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \varrho h \frac{\partial^2 u_0}{\partial t^2} \tag{27}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = \varrho h \frac{\partial^2 v_0}{\partial t^2} \tag{28}$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = \varrho h \frac{\partial^2 w_0}{\partial t^2} - q(x,y,t) \tag{29}$$

where

$$\varrho = \sum_{k=1}^{N} \frac{\varrho_k h_k}{h} \tag{30}$$

If Equations (17) and (18) were substituted into the equilibrium equations in terms of the midplane displacements u_0, v_0 and w_0, and the symbols u, v and w were used for brevity, then the equations of motion become

$$\begin{aligned}
&\left[A_{11} \frac{\partial^2 u}{\partial x^2} + 2A_{16} \frac{\partial^2 u}{\partial x \partial y} + A_{66} \frac{\partial^2 u}{\partial y^2} + A_{16} \frac{\partial^2 v}{\partial x^2} \right. \\
&\left. \quad + (A_{12} + A_{66}) \frac{\partial^2 v}{\partial x \partial y} + A_{26} \frac{\partial^2 v}{\partial y^2}\right] \\
&\quad - \left[B_{11} \frac{\partial^3 w}{\partial x^3} + 3B_{16} \frac{\partial^3 w}{\partial x^2 \partial y} + (B_{12} + 2B_{66}) \frac{\partial^3 w}{\partial x \partial y^2} + B_{26} \frac{\partial^3 w}{\partial y^3}\right] \\
&\quad = \varrho h \frac{\partial^2 u}{\partial t^2} + \left[(E_{11}d_{31} + E_{12}d_{32} + E_{16}d_{36}) \frac{\partial R}{\partial x} \right. \\
&\left. \quad + (E_{16}d_{31} + E_{26}d_{32} + E_{66}d_{36}) \frac{\partial R}{\partial y}\right]
\end{aligned} \tag{31}$$

$$\begin{aligned}
&\left[A_{16} \frac{\partial^2 u}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 u}{\partial x \partial y} + A_{26} \frac{\partial^2 u}{\partial y^2} \right. \\
&\left. \quad + A_{66} \frac{\partial^2 v}{\partial x^2} + 2A_{26} \frac{\partial^2 v}{\partial x \partial y} + A_{22} \frac{\partial^2 v}{\partial y^2}\right] \\
&\quad - \left[B_{16} \frac{\partial^3 w}{\partial x^3} + (B_{12} + 2B_{66}) \frac{\partial^3 w}{\partial x^2 \partial y} + 3B_{26} \frac{\partial^3 w}{\partial x \partial y^2} + B_{22} \frac{\partial^3 w}{\partial y^3}\right] \\
&\quad = \varrho h \frac{\partial^2 v}{\partial t^2} + \left[(E_{16}d_{31} + E_{26}d_{32} + E_{66}d_{36}) \frac{\partial R}{\partial x} \right. \\
&\left. \quad + (E_{21}d_{31} + E_{22}d_{32} + E_{26}d_{36}) \frac{\partial R}{\partial y}\right]
\end{aligned} \tag{32}$$

$$\left[B_{11}\frac{\partial^3 u}{\partial x^3} + 3B_{16}\frac{\partial^3 u}{\partial x^2\partial y} + (B_{12} + 2B_{66})\frac{\partial^3 u}{\partial x\partial y^2} + B_{26}\frac{\partial^3 u}{\partial y^3}\right.$$

$$\left. + B_{16}\frac{\partial^3 v}{\partial x^3} + (B_{12} + 2B_{66})\frac{\partial^3 v}{\partial x^2\partial y} + 3B_{26}\frac{\partial^3 v}{\partial x\partial y^2} + B_{22}\frac{\partial^3 v}{\partial y^3}\right]$$

$$- \left[D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3\partial y} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}\right.$$

$$\left. + 4D_{26}\frac{\partial^4 w}{\partial x\partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4}\right]$$

$$= \varrho h\frac{\partial^2 w}{\partial t^2} - q(x,y,t) + (F_{11}d_{31} + F_{12}d_{32} + F_{16}d_{36})\frac{\partial^2 R}{\partial x^2}$$

$$+ (F_{21}d_{31} + F_{22}d_{32} + F_{26}d_{36})\frac{\partial^2 R}{\partial y^2}$$

$$+ 2(F_{16}d_{31} + F_{26}d_{32} + F_{66}d_{36})\frac{\partial^2 R}{\partial x\partial y} \tag{33}$$

Equations (31–33) are the equations of motion in terms of the midplane displacement. The last two terms of the right-hand side of Equations (31) and (32) include the $\partial R/\partial x$ and $\partial R/\partial y$ which can be recognized as vertical line forces illustrated in Table 1. The third and fourth terms of the right-hand side of Equation (33) include $\partial^2 R/\partial x^2$ and $\partial^2 R/\partial y^2$ respectively which can be recognized as line moments along the edges of the actuator patch illustrated in Table 1. Additionally, the fifth term of the right-hand side of Equation (33) includes $\partial^2 R/\partial x\partial y$ which can be recognized as the concentrated forces at the corners of the actuator patch also shown in Table 1. It is noted that these concentrated forces result in laminate twisting. To solve the equations of motion, boundary conditions need to be specified. The general boundary conditions can be categorized as follows (Whitney, 1987):

1. Simply supported: $N_n = N_{ns} = w = M_n = 0$
2. Hinged-free in the normal direction: $N_n = u_{ns} = w = M_n = 0$
3. Hinged-free in the tangential direction: $u_n = N_{ns} = w = M_n = 0$
4. Clamped: $u_n = u_s = w = M_n = 0$
5. Free: $N_s = N_{ns} = w = \partial M_{ns}/\partial s + Q_n = 0$

EXAMPLES

One Isotropic Layer with Two Piezoceramic Patches (Pure Bending)

Considered a simply supported, isotropic lamina with two actuator patches

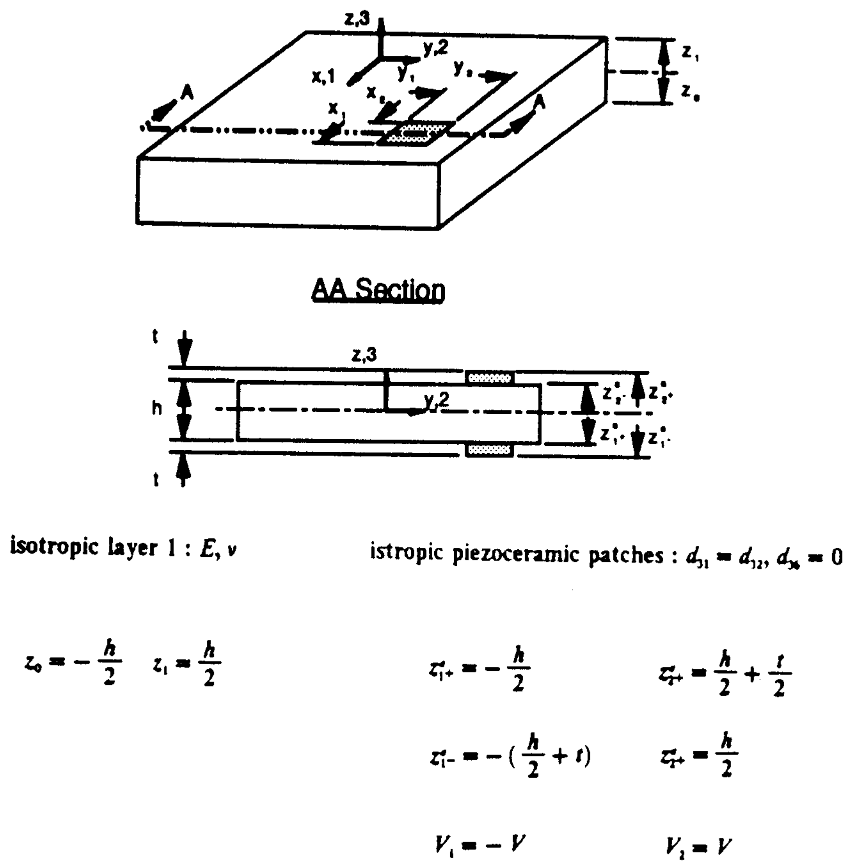


Figure 3. *Simply supported lamina with two piezoceramic patches (pure bending).*

(piezoceramic) bonded to the top and bottom surfaces of the plate as shown in Figure 3. The material properties for the isotropic layer are given as:

$$[\bar{Q}] = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix} \tag{34}$$

The material properties for the actuator patch are given as:

$$[\bar{Q}^a] = \begin{bmatrix} \frac{E_a}{1-\nu_a^2} & \frac{\nu_a E_a}{1-\nu_a^2} & 0 \\ \frac{\nu_a E_a}{1-\nu_a^2} & \frac{E_a}{1-\nu_a^2} & 0 \\ 0 & 0 & \frac{E_a}{2(1+\nu_a)} \end{bmatrix} \tag{35}$$

where E, ν, E_a and ν_a are engineering constants for the isotropic layer and actuator patches. Table 2 shows the typical physical properties of G-1195 piezoceramic patch (Piezo Systems, 1987). It is noted that the laminate is symmetry. If the applied voltages of the piezoceramic patches were out of phase, i.e., $V_1 = -V_2 = -V$, then the following relations result

$$A_{ij} = \bar{Q}_{ij} h \tag{36}$$

$$B_{ij} = 0 \tag{37}$$

$$D_{ij} = \frac{h^3}{12}\bar{Q}_{ij} \tag{38}$$

$$E_{ij} = 0 \tag{39}$$

$$F_{ij} = \bar{Q}^a_{ij} V(h+t) \tag{40}$$

It is noted that $(h + t)$ is the distance between the two piezoceramic patches. The equations of motion can be reduced to

$$A_{11}\frac{\partial^2 u}{\partial x^2} + A_{66}\frac{\partial^2 u}{\partial y^2} + (A_{12}+A_{66})\frac{\partial^2 v}{\partial x \partial y} = \varrho h \frac{\partial^2 u}{\partial t^2} \tag{41}$$

$$(A_{12}+A_{66})\frac{\partial^2 u}{\partial x \partial y} + A_{66}\frac{\partial^2 v}{\partial x^2} + A_{22}\frac{\partial^2 v}{\partial y^2} = \varrho h \frac{\partial^2 v}{\partial t^2} \tag{42}$$

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12}+D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} + \varrho h \frac{\partial^2 w}{\partial t^2}$$
$$= q(x,y,t) + \left[(F_{11}d_{31} + F_{12}d_{32})\frac{\partial^2 R}{\partial x^2} + (F_{21}d_{31} + F_{22}d_{32})\frac{\partial^2 R}{\partial y^2}\right] \tag{43}$$

Table 2. Physical properties of G-1195 piezoceramic patch.

$d_{31} = d_{32} = 1.66 \times 10^{-12} \left(\frac{\text{m}}{\text{volt}}\right)$	$d_{36} = 0$
$\varrho_a = 7650 \left(\frac{\text{kg}}{\text{m}^3}\right)$	$E_a = 6.3 \times 10^{10} \left(\frac{\text{N}}{\text{m}^2}\right)$
$t = 1.905$ (mm)	$\nu_a = 0.28$

One can observe that Equations (41) and (42) are coupled without any actuator effects, and Equation (43) with actuator effects is independent. Both Equations (41) and (42) are known as the stretching problem, and Equation (43) is known as the bending problem. These equations of motion show that the two actuator patches only induce laminate bending, and the equivalent external forces excited by the two actuator patches are the distributed line moments along the four edges of actuator patches. This result coincides with that of Dimitriadis et al. (1989). For comparison to their result, the bending equation substituted by D_{ij} and F_{ij} can be written as:

$$D\nabla^4 w + \varrho h \frac{\partial^2 w}{\partial t^2} = C_0' \Lambda \left(\frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2}\right) \tag{44}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

$$C_0' = t(h+t)\frac{E_a}{1-\nu_a}$$

$$\Lambda = \frac{d_{31}}{t} V$$

The corresponding equation derived from the work of Dimitriadis et al. (1989) is

$$D\nabla^4 w + \varrho h \frac{\partial^2 w}{\partial t^2} = C_0 \Lambda \left(\frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2}\right) \tag{45}$$

where

$$C_0 = -E\frac{1+\nu_a}{1-\nu}\frac{P}{1+\nu-(1+\nu_a)P}\frac{2}{3}\left(\frac{h}{2}\right)^2 \tag{46}$$

$$P = -\frac{E_a}{E}\frac{1-\nu^2}{1-\nu_a^2}\frac{3t\left(\frac{h}{2}\right)(h+t)}{2\left[\left(\frac{h}{2}\right)^3 + t^3\right] + 3\left(\frac{h}{2}\right)t^2} \tag{47}$$

The main difference of the current work from that of Dimitriadis et al. (1989) is the material constant C_0' and C_0. As C_0' is a function of E_a, ν_a, h and t, C_0 is a function of E, ν, E_a, ν_a, h and t. One can easily show that if $h \gg t$, and $E \gg E_a$, then

$$C_0' = C_0 = th \frac{E_a}{1 - \nu_a} \tag{48}$$

To explain this coincidence, we can see that the current approach assumes the expansion and contraction of piezoceramic patches with free constraint; however, Dimitriadis et al. (1989) included the interactions between the plate and piezoceramic patch upon the assumption of spherical bending. When the thickness and modulus ratios are large, the interaction effects between the plate and actuator patches is not significant. Figures 4 and 5 show the comparison of these two models by illustrating the ratio C_0/C_0' as a function of thickness and modulus ratios, respectively.

One Isotropic Layer with Two Piezoceramic Patches (Pure Extension)

In this case study a similar plate was considered as that in the previous case study except that the applied voltages $V_1 = V_2 = V$. The $[A]$, $[B]$ and $[D]$ matrices are unchanged, and

$$E_{ij} = 2\bar{Q}^a_{ij} V \tag{49}$$

$$F_{ij} = 0 \tag{50}$$

The equations of motion can be reduced to

$$A_{11} \frac{\partial^2 u}{\partial x^2} + A_{66} \frac{\partial^2 u}{\partial y^2} + (A_{12} + A_{66}) \frac{\partial^2 v}{\partial x \partial y} = \varrho h \frac{\partial^2 u}{\partial t^2} + (E_{11} d_{31} + E_{12} d_{32}) \frac{\partial R}{\partial x} \tag{51}$$

$$(A_{12} + A_{66}) \frac{\partial^2 u}{\partial x \partial y} + A_{66} \frac{\partial^2 v}{\partial x^2} + A_{22} \frac{\partial^2 v}{\partial y^2} = \varrho h \frac{\partial^2 v}{\partial t^2} + (E_{21} d_{31} + E_{22} d_{32}) \frac{\partial R}{\partial y} \tag{52}$$

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^2 w}{\partial y^4} + \varrho h \frac{\partial^2 w}{\partial t^2} = q(x,y,t) \tag{53}$$

One can observe that Equations (51) and (52) are coupled, and the piezoelectric

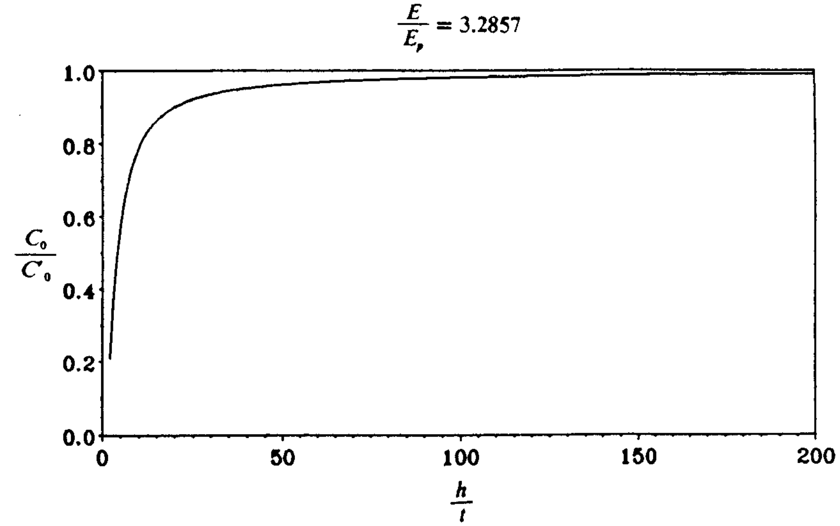


Figure 4. Typical results of C_0/C_0' by varying thickness ratio.

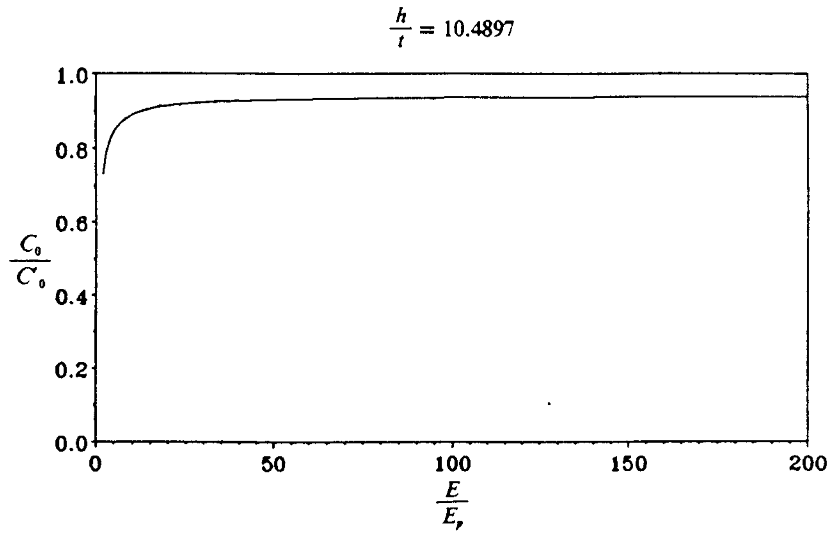


Figure 5. Typical results of C_0/C_0' by varying modulus ratio.

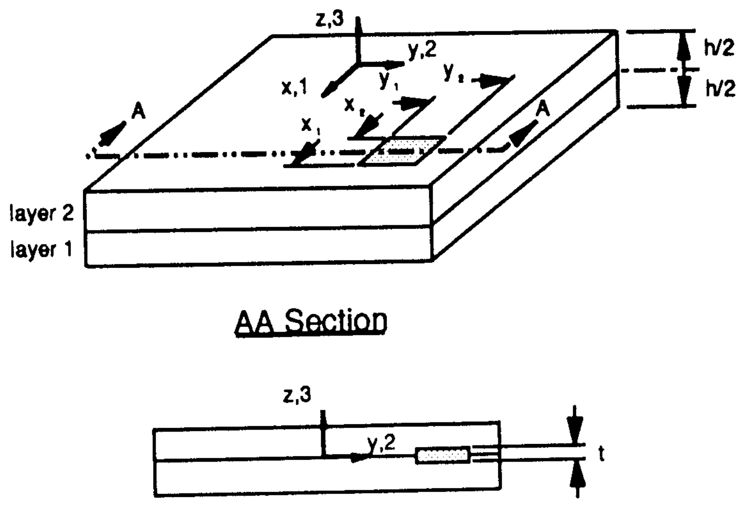


Figure 6. Simply supported two-layer laminate ($\theta°/-\theta°$) with one piezoceramic patch.

effects are included; however, Equation (53) is independent without any actuation terms. Therefore, this example results in pure extension rather than the pure bending of the previous example. In Equations (51) and (52), the terms involved $\partial R/\partial x$ and $\partial R/\partial y$ which represent the vertical forces along the edges are the equivalent external forces induced by the piezoceramic patches.

Two-Layered ($\theta°/-\theta°$) Laminate with One Piezoceramic Patch

Considered a two-layer, angle-ply laminate with one piezoceramic patch embedded within the laminate and between the two laminae. The laminated plate is illustrated in Figure 6. The following relations can be found

$$A_{ij} = \frac{h}{2}\,[(\overline{Q}_{ij})_1 + (\overline{Q}_{ij})_2] \tag{54}$$

$$B_{ij} = \frac{h^2}{8}\,[(\overline{Q}_{ij})_1 - (\overline{Q}_{ij})_2] \tag{55}$$

$$D_{ij} = \frac{h^3}{24}\,[(\overline{Q}_{ij})_1 + (\overline{Q}_{ij})_2] \tag{56}$$

$$E_{ij} = \overline{Q}^a_{ij} V \tag{57}$$

$$F_{ij} = 0 \tag{58}$$

It is noted that actuator bending-twisting stiffnesses, F_{ij}, are zero. That is to say, there is no bending and twisting induced by the pipezoceramic patch; however, bending and extension are coupled due to the non-symmetric, angle-ply laminate. This coupling can be seen from the following equations of motion.

$$\begin{aligned}
&\left[A_{11}\frac{\partial^2 u}{\partial x^2} + 2A_{16}\frac{\partial^2 u}{\partial x\partial y} + A_{66}\frac{\partial^2 u}{\partial y^2} + A_{16}\frac{\partial^2 v}{\partial x^2}\right.\\
&\qquad\left. + (A_{12} + A_{66})\frac{\partial^2 v}{\partial x\partial y} + A_{26}\frac{\partial^2 v}{\partial y^2}\right]\\
&\qquad - \left[B_{11}\frac{\partial^3 w}{\partial x^3} + 3B_{16}\frac{\partial^3 w}{\partial x^2\partial y} + (B_{12} + 2B_{66})\frac{\partial^3 w}{\partial x\partial y^2} + B_{26}\frac{\partial^3 w}{\partial y^3}\right]\\
&\qquad = \varrho h\frac{\partial^2 u}{\partial t^2} + (E_{11}d_{31} + E_{12}d_{32})\frac{\partial R}{\partial x}
\end{aligned} \tag{59}$$

$$\begin{aligned}
&\left[A_{16}\frac{\partial^2 u}{\partial x^2} + (A_{12} + A_{66})\frac{\partial^2 u}{\partial x\partial y} + A_{26}\frac{\partial^2 u}{\partial y^2}\right.\\
&\qquad\left. + A_{66}\frac{\partial^2 v}{\partial x^2} + 2A_{26}\frac{\partial^2 v}{\partial x\partial y} + A_{22}\frac{\partial^2 v}{\partial y^2}\right]\\
&\qquad - \left[B_{16}\frac{\partial^3 w}{\partial x^3} + (B_{12} + 2B_{66})\frac{\partial^3 w}{\partial x^2\partial y} + 3B_{26}\frac{\partial^3 w}{\partial x\partial y^2} + B_{22}\frac{\partial^3 w}{\partial y^3}\right]\\
&\qquad = \varrho h\frac{\partial^2 v}{\partial t^2} + (E_{21}d_{31} + E_{22}d_{32})\frac{\partial R}{\partial y}
\end{aligned} \tag{60}$$

$$\begin{aligned}
&\left[B_{11}\frac{\partial^3 u}{\partial x^3} + 3B_{16}\frac{\partial^3 u}{\partial x^2\partial y} + (B_{12} + 2B_{66})\frac{\partial^3 u}{\partial x\partial y^2} + B_{26}\frac{\partial^3 u}{\partial y^3}\right.\\
&\qquad\left. + B_{16}\frac{\partial^3 v}{\partial x^3} + (B_{12} + 2B_{66})\frac{\partial^3 v}{\partial x^2\partial y} + 3B_{26}\frac{\partial^3 v}{\partial x\partial y^2} + B_{22}\frac{\partial^3 v}{\partial y^3}\right]\\
&\qquad - \left[D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3\partial y} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}\right.\\
&\qquad\left. + 4D_{26}\frac{\partial^4 w}{\partial x\partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4}\right]\\
&\qquad = \varrho h\frac{\partial^2 w}{\partial t^2} - q(xy,t)
\end{aligned} \tag{61}$$

All three equations, Equations (59–61), are coupled. The actuator patch results in two equivalent external forces which are the last terms of Equation (59) and (60), respectively. To calculate the responses, Equations (59–61) must be solved simultaneously.

CONCLUSION

This paper presents a model based upon CLPT theory for laminated plates with embedded finite-length, multiple-layer induced strain actuator patches. A generalized location function which was expressed with the Heaviside function was introduced to represent the size and location of the spatially distributed actuator patches. The equivalent external forces and moments induced by the actuator patches can be determined so that the responses can be easily solved. A few case studies, such as piezoceramic patches bonded to the surface and embedded within a laminate, were illustrated for pure bending and pure extension states. In addition, an example of a two-layer, non-symmetric, angle-ply laminate with one actuator patch embedded within the middle showed that piezoceramic patch can be used to induce bending and extension coupling.

NOMENCLATURE

$[A]$ extensional stiffnesses
$[B]$ bending-twist coupling stiffnesses
C_0 material constant for one isotropic layer and two piezoceramic patches (Dimitriadis et al., 1989)
C_0' material constant for one isotropic layer and two piezoceramic patches (Wang and Rogers, 1989)
$[D]$ bending stiffnesses
D flexural rigidity
$\{d\}$ piezoelectric strain coefficient vectors
$(d_{ij})_k$ piezoelectric strain coefficient for the k-th layer actuator patch
$[E]$ actuator extensional stiffnesses
E Young's modulus
$[F]$ actuator bending-twisting stiffnesses
$H(x - x_0)$ Heaviside function
h thickness of laminate
h_k thickness of the k-th layer lamina
$\{M\}$ resultant moment vector
M_x, M_y, M_{xy} resultant moments
m number of actuator patches
$\{N\}$ resultant force vector
N_x, N_y, N_{xy} resultant forces
n number of laminae
$[\bar{Q}]$ material properties matrix in (x,y,z) coordinates
$[Q]$ material properties matrix in (1,2,3) coordinates
$\bar{Q}_{ij}$ material properties in (x,y,z) coordinates
Q_{ij} material properties in (1,2,3) coordinates
Q_n transverse shear in the normal direction
$q(x,y,t)$ transverse load
$R_k(x,y)$ generalized location function of the k-th actuator patch
t thickness of actuator patch
V_k voltage applied to the k-layer actuator patch
$(x_i)_k, (y_i)_k$ position coordinates of actuator patches
(x,y,z) the laminated plate coordinates
z_k thickness coordinate of the k-th layer
$z_k^a, z_{k'}^a$ coordinate of actuator patch in z-direction
(1,2,3) the principal material coordinates for a lamina
γ_{xy} shear strain
γ_{xy}^m mechanical shear strain
∇ divergence operator
$\delta(x - x_0)$ delta function
$\delta'(x - x_0)$ first derivative of delta function
$\{\epsilon\}$ total strain vector in (x,y,z) coordinates
$\{\bar{\epsilon}\}$ total strain vector in (1,2,3) coordinates
$\{\epsilon^m\}$ mechanical strain vector
$\{\epsilon^0\}$ midplane mechanical strain vector
$\epsilon_x^m, \epsilon_y^m$ mechanical normal strain
$\{\kappa\}$ midplane mechanical curvature vector
$\kappa_x, \kappa_y, \kappa_{xy}$ midplane mechanical curvature vector
$\{\Lambda\}$ actuator strain vector
Λ_x, Λ_y actuator normal strain
Λ_{xy} actuator shear strain
ν Poisson ratio
ϱ equivalent density of laminate
ϱ_k density of the k-th layer lamina
$\{\sigma\}$ stress vector in (x,y,z) coordinate
$\{\bar{\sigma}\}$ stress vector in (1,2,3) coordinate
σ_x, σ_y normal stress
τ_{xy} shear stress

Superscript

a actuator
m mechanical
0 midplane

Subscript

a actuator
k k-th layer
n normal direction
s tangential direction

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